

SEMICLASSICAL L^2 ESTIMATES FOR RESTRICTIONS OF THE QUANTISATION OF NORMAL VELOCITY TO INTERIOR HYPERSURFACES

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ABSTRACT. We seek to extend recent work by Christianson-Hassell-Toth [2] on restrictions of Neumann data of Laplacian eigenfunctions to interior hypersurfaces. In the semiclassical regime the appropriate generalisation is to study the restrictions of the function $v = \nu(x, hD)u$ where $\nu(x, hD)$ is the operator defined by quantising the normal velocity observable. For the Laplacian $\nu(x, hD) = \frac{1}{2}hD\nu$ where ν is the normal to the hypersurface. We find that $\|\nu(x, hD)u\|_{L^2(H)} \lesssim \|u\|_{L^2(M)}$ provided u is an $O_{L^2}(h)$ quasimode of the semiclassical pseudodifferential operator $p(x, hD)$.

Consider an eigenfunction u of the Laplace-Beltrami operator, that is

$$-\Delta u = \lambda^2 u$$

In [3] Hassell and Tao showed that for Dirichlet eigenfunctions

$$(1) \quad \|\lambda^{-1} \partial_\nu u\|_{L^2(\partial M)} \lesssim \|u\|_{L^2(M)}$$

where ν is the normal to the the boundary ∂M . More recently Christianson, Hassell and Toth [2] obtain the equivalent estimate for interior hypersurfaces, that is

$$\|\lambda^{-1} \partial_\nu u\|_{L^2(H)} \lesssim \|u\|_{L^2(M)}$$

This estimate should be seen as a statement of non-concentration. Note that by Burq-Gérard-Tvetkov [1] we know that there are sharp examples u such that

$$\|u\|_{L^2(H)} \lesssim \lambda^{1/4} \|u\|_{L^2(M)}.$$

However these eigenfunctions have comparatively small, $O(\lambda^{1/2})$, normal derivative so for this class of examples

$$\|\lambda^{-1} \partial_\nu u\|_{L^2(H)} \lesssim \lambda^{-1/4} \|u\|_{L^2(M)}$$

In this paper we move the problem into a semiclassical setting to gain some intuition from quantum-classical correspondence principles.

For a smooth symbol $p(x, \xi)$ understood to represent the total (conserved) energy of a system we define the classical flow on phase space by

$$(2) \quad \begin{cases} \dot{x}_i(t) = \partial_{\xi_i} p(x, \xi) \\ \dot{\xi}_i(t) = -\partial_{x_i} p(x, \xi). \end{cases}$$

The simplest example of such a system is that of free particle motion given by the symbol $p(x, \xi) = |\xi|_g^2$. In the classical setting observables are given by symbols

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$q(x, \xi)$ defined on phase space. We can then move to the semiclassical setting by quantising these symbols to obtain semiclassical pseudodifferential operators

$$q(x, hD)u = Op(q(x, \xi))u = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}\langle x-y, \xi \rangle} q(x, \xi) u(y) d\xi dy.$$

The Laplace operator is obtained by quantising the symbol $p(x, \xi) = |\xi|_g^2$ and therefore is the quantisation of the energy observable of free particle motion. For a hypersurface $H = \{x \mid x_1 = 0\}$ with $\lambda^{-1} = h$ the operator $\lambda^{-1}\partial_{x_1}$ is (up to constants) the quantisation of the symbol $\partial_{\xi_1}p(x, \xi)$ or the quantisation of the normal velocity observable. A productive intuition in this area is to consider u as being comprised of small wave packets, localised in phase space, that propagate according to the classical flow. Therefore we expect to see concentration only when packets spend a long time trapped near the hypersurface. For free particle motion such trajectories must therefore have small normal velocity and so packets tracking along them are not expected to make large contributions to $hD_{x_1}u$. The large contributions come from packets moving along trajectories with $O(1)$ normal velocity, however such packets spend little time near the hypersurface and are known not to concentrate. We can of course define a classical flow given by (2) for any symbol $p(x, \xi)$ so in the semiclassical setting we should aim to prove that the operator given by quantising normal velocity does not give rise to concentration.

Accordingly we define the symbol $\nu(x, \xi)$ by

$$\nu(x, \xi) = \partial_{\xi_1}p(x, \xi)$$

and aim to show that

$$\|\nu(x, hD)u\|_{L^2(H)} \lesssim \|u\|_{L^2(M)}$$

under the condition that u is semiclassically localised.

Definition 0.1. *We say u is semiclassically localised if there exists $\chi \in C_c(T^*M)$ such that*

$$u = \chi(x, hD)u + O_S(h^\infty).$$

where S is the space of Schwartz functions.

The main theorem of this paper is therefore Theorem 0.2.

Theorem 0.2. *Let (M, g) be a smooth Riemannian manifold of dimension n and let H be a smooth embedded interior hypersurface given in local coordinates by $\{x \mid x_1 = 0\}$. Suppose $u(h)$ is a family of semiclassically localised, L^2 normalised functions that satisfy $p(x, hD)u = O_{L^2}(h)$ where $p(x, hD)$ is a semiclassical pseudodifferential operator with smooth symbol $p(x, \xi)$. Then*

$$\|\nu(x, hD)u\|_{L^2(H)} \lesssim 1$$

for $\nu(x, hD)$ the semiclassical pseudodifferential operator with symbol

$$\nu(x, \xi) = \partial_{\xi_1}p(x, \xi)$$

Eigenfunctions of the Laplacian can be written as solutions to the semiclassical equation $p(x, hD)u = 0$ where $p(x, hD)$ is the semiclassical pseudodifferential operator with symbol $p(x, \xi) = |\xi|_g^2 - 1$ and therefore fall under the scope of Theorem 0.2. This allows us to reproduce bounds on the Neumann data for interior hypersurfaces.

Corollary 0.3. *Let (M, g) be a smooth Riemannian manifold and H a smooth embedded interior hypersurface with normal $\nu(x)$ if u is an L^2 normalised Laplacian eigenfunction*

$$\Delta u = -\lambda^2 u$$

then

$$\|\lambda^{-1} \partial_\nu u\|_{L^2(H)} \lesssim 1.$$

Proof. Working in Fermi normal coordinates in a small tubular neighbourhood of the hypersurface we may write

$$-h^2 \Delta_g - 1 = p(x, hD)$$

where $p(x, hD)$ has principal symbol

$$p(x, \xi) = \xi_1^2 + q(x, \xi')$$

and therefore

$$\nu(x, \xi) = 2\xi_1$$

$$\nu(x, hD) = 2hD_\nu$$

and so by Theorem 0.2 with $h = \lambda^{-1}$

$$\|\lambda^{-1} \partial_\nu u\|_{L^2(H)} \lesssim 1$$

as required. □

1. PROOF OF THEOREM 0.2

We will prove Theorem 0.2 by dyadically decomposing $\nu(x, hD)u$. In what follows we write a point in M as $x = (x_1, x')$ and the dual variables as $\xi = (\xi_1, \xi')$. As u is semiclassically localised we may write

$$u = \chi(x, hD)u + O(h^\infty).$$

Then let $\zeta_0, \zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be smooth cut off functions such that ζ_0 is supported in $r < 2$ and ζ is supported in $1 < r < 4$. Further suppose that on the support of $\chi(x, \xi)$

$$1 = \zeta_0(h^{-1/2}r) + \sum_{j=1}^J \zeta(2^{-j}h^{-1/2}r) \quad \text{where } 2^{-J} = h^{1/2}.$$

Now we define the symbols $\nu_j(x, \xi)$ by

$$(3) \quad \nu_0(x, \xi) = \zeta_0(h^{-1/2}|\nu(x, \xi)|)\nu(x, \xi)$$

and

$$(4) \quad \nu_j(x, \xi) = \zeta(2^{-j}h^{-1/2}|\nu(x, \xi)|)\nu(x, \xi) \quad j \geq 1$$

The associated operators $\nu_j(x, hD)$ are given by the quantisation formula, that is

$$(5) \quad \nu_j(x, hD) = Op(\nu_j(x, hD)).$$

We will show that

$$\|\nu_j(x, hD)u\|_{L^2(H)} \lesssim 2^{j/2} h^{1/4} \|u\|_{L^2(M)}$$

and therefore that

$$\begin{aligned} \left\| \sum_{j=1}^J \nu(x, hD) u \right\|_{L^2(H)} &\lesssim \|u\|_{L^2(M)} \sum_{j=1}^J 2^{j/2} h^{1/4} \\ &\lesssim \|u\|_{L^2} \sum_{j=0}^{\infty} 2^{-j/2} \lesssim \|u\|_{L^2(M)}. \end{aligned}$$

Note that since u is a quasimode of $p(x, hD)$ and

$$p(x, hD) \nu_J(x, hD) = \nu_J(x, hD) p(x, hD) u + O_{L^2}(h),$$

$\nu(x, hD) u$ is also an $O_{L^2}(h)$ quasimode of $p(x, hD)$. Since $2^J h^{1/2} = O(1)$ we have

$$|\partial_{\xi_1} p(x, \xi)| > c > 0.$$

This case was treated in [4] giving

$$(6) \quad \|\nu_J(x, hD) u\|_{L^2(H)} \lesssim \|\nu_J(x, hD) u\|_{L^2(M)} \lesssim \|u\|_{L^2(M)}.$$

Estimate (6) follows from the fact that in this case we may factorise the symbol $p(x, \xi)$ as

$$p(x, \xi) = e(x, \xi)(\xi_1 - a(x, \xi'))$$

where $|e(x, \xi)| > c > 0$. This means that $\nu(x, \xi) u$ is an $O_{L^2}(h)$ quasimode of the operator $hD_{x_1} - a(x, hD)$. Associating x_1 with t we see this as an evolution operator and with $v_J = \nu_J(x, hD) u$ write

$$(7) \quad v_J(x) = U(t) v_J(0, x') + \frac{i}{h} \int_0^t U(t-s) E[v_J] ds$$

where $U(t)$ satisfies

$$\begin{cases} (hD_t - a(x, hD_{x'})) U(t) = 0 \\ U(0) = \text{Id} \end{cases}$$

and $E[v_J]$ is the quasimode error

$$E[v_J] = (hD_{x_1} - a(x, hD_{x'})) v_J.$$

Now since (7) is true for all $t \in [0, 1]$ we average over that time period to obtain (6). gives the expected estimate for $j = J$. For $j < J$ we want to use the same idea however we cannot simply factorise as is because we cannot assume that $|\nabla_{\xi'} p(x, \xi)| \lesssim 2^j h^{1/2}$. First we add an additional variable (which we will later integrate out) and consider u as a time independent solution of the evolution equation

$$(8) \quad (hD_t - 2^{-j} h^{-1/2} p(x, hD)) v(t, x) = E_j[u]$$

where

$$E_j[u] = 2^{-j} h^{-1/2} p(x, hD) u = O_{L^2}(2^{-j} h^{1/2})$$

Therefore using Duhamel's principle we can write

$$(9) \quad u(x) = U_j(t) u + \frac{i}{h} \int_0^t U_j(t-s) E_j[u] ds$$

where $U_j(t)$ is the evolution operator satisfying

$$(10) \quad \begin{cases} (hD_t - 2^{-j} h^{-1/2} p(x, hD)) U_j(t) = 0 \\ U_j(0) = \text{Id}. \end{cases}$$

We now average (9) over time periods of order T . That is let $\chi(t)$ be a smooth function supported in $[-\epsilon, \epsilon]$ such that

$$\int \chi(t) = 1.$$

Then we have that

$$(11) \quad u(x) = \frac{1}{T} \int \chi\left(\frac{t}{T}\right) U_j(t) u dt + \frac{i}{hT} \iint_0^t \chi\left(\frac{t}{T}\right) U_j(t-s) E_j[u] ds dt.$$

So we obtain

$$(12) \quad \nu_j(x, hD)u = \frac{1}{T} \int \chi\left(\frac{t}{T}\right) \nu_j(x, hD) U_j(t) u dt + \frac{i}{hT} \iint_0^t \chi\left(\frac{t}{T}\right) \nu_j(x, hD) U_j(t-s) E[u] ds dt.$$

Since u is an $O_{L^2}(h)$ quasimode of $p(x, hD)$ it is an $O_{L^2}(2^{-j}h^{1/2})$ quasimode of $2^{-j}h^{-1/2}p(x, hD)$ so we take $T = 2^j h^{1/2}$. Therefore if we set

$$(13) \quad W_{j,1}u = 2^{-j}h^{-1/2} \int \chi\left(2^{-j}h^{-1/2}t\right) \nu_j(x, hD) U(t) u dt$$

and

$$(14) \quad W_{j,2}u = 2^{-j}h^{-3/2} \iint_0^t \chi\left(2^{-j}h^{-1/2}t\right) \nu_j(x, hD) U(t-s) E[u] ds dt$$

we need only show that

$$\|W_{j,i}u\|_{L^2(H)} \lesssim 2^{j/2}h^{1/4} \|u\|_{L^2(M)} \quad i = 1, 2.$$

Proposition 1.1. *Let $j \geq 1$ $W_{j,1}$ and $W_{j,2}$ be given by (13) and (14) respectively then*

$$(15) \quad \|W_{j,i}u\|_{L^2(H)} \lesssim 2^{j/2}h^{1/4} \|u\|_{L^2(M)} \quad i = 1, 2.$$

Proof. We use a semiclassical parameterix construction to represent $U_j(t)$, (see Zworski [5] for proof). We have that if $U_j(t)$ satisfies (10)

$$(16) \quad U_j(t)v(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(\phi(t,x,\xi) - \langle y, \xi \rangle)} b(t, x, \xi, h) v(y) dy d\xi + O(h^\infty)$$

where

$$\partial_t \phi(t, x, \xi) + p(x, \partial_x \phi(t, x, \xi)) = 0, \quad \phi(0, x, \xi) = x \cdot \xi$$

$$b(t, x, \xi, h) \in C_c^\infty(\mathbb{R} \times T^*\mathbb{R}^n \times \mathbb{R}) \quad E(t) = O(h^\infty) : S' \rightarrow S.$$

We are then able to extract decay from the oscillatory nature of this integral.

We begin with $W_{j,1}$, for $j \geq 1$ we have

$$\begin{aligned} W_{j,1}v &= \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t,x',\xi) - \langle y, \xi \rangle)} \zeta(2^{-j}h^{-1/2}|\nu(x, \nabla_x \phi)|) (b(t, x, \xi, h) \\ &\quad \times \chi\left(2^{-j}h^{1/2}t\right) v(y) dy d\xi dt + \tilde{E}_j v \end{aligned}$$

where

$$(17) \quad \tilde{E}_j v = \frac{h}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t,x,\xi) - \langle y, \xi \rangle)} \tilde{b}_j(t, x, \xi, h) \chi(t) v(y) dy d\xi dt + O(h^\infty)$$

$$|D^\alpha b_j(t, x, \xi; h)| \leq C_\alpha 2^{-j|\alpha|} h^{-|\alpha|/2}.$$

The extra factor of h in the error term will allow us to treat it simply and we defer that proof to Lemma 1.3 where we show that

$$\|E_j v\|_{L^2(H)} \lesssim h^{1/2} \|v\|_{L^2(M)}$$

and therefore can be ignored. We therefore focus on the first term which in a slight abuse of notation we will continue to refer to as $W_{j,1}$. Note that since $x_1 = 0$ there is a critical point in ξ_1 when

$$y_1 = O(t) = O(2^{-j} h^{1/2}).$$

so integration by parts gives us

$$(18) \quad W_{j,1} v = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t,x',\xi) - \langle y, \xi \rangle)} \zeta(2^{-j} h^{-1/2} |\nu(x, \nabla_x \phi)|) b(t, x, \xi, h) \\ \times \chi\left(2^{-j} h^{1/2} t\right) \left(1 + \frac{|y_1|}{2^j h^{1/2}}\right)^{-N} v(y) dy d\xi dt.$$

Now we want to use stationary phase to calculate the (t, ξ_1) integral however first we must scale out the ξ' variables (which essentially reduces the problem to a one dimensional one). We introduce the scalings

$$x' \rightarrow 2^j h^{1/2} x' \quad y' \rightarrow 2^j h^{1/2} \quad \xi' \rightarrow 2^j h^{1/2}$$

and so we have

$$W_{j,1} v(0, 2^j h^{1/2} x') = 2^{2j(n-1)} h^{-1} \int e^{\frac{i}{h}(\phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') - y_1 \xi_1 - 2^{2j} h \langle y', \xi' \rangle)} B_j(t, x', \xi) \\ \times \left(1 + \frac{|y_1|}{2^j h^{1/2}}\right)^{-N} v(y_1, 2^j h^{1/2} y') dy d\xi dt$$

where

$$B_j(t, x', \xi) = \zeta(2^{-j} h^{-1/2} |\nu(x, \nabla_x \phi)|) b(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi', h) \chi\left(2^{-j} h^{1/2} t\right)$$

Note that as we are mapping $L_{x'}^2 \rightarrow L_{x'}^2$ we need only prove that the operator

$$(19) \quad T_{j,1} w = 2^{2j(n-1)} h^{-1} \int e^{\frac{i}{h}(\phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') - y_1 \xi_1 - 2^{2j} h \langle y', \xi' \rangle)} B_j(t, x', \xi) \\ \times \left(1 + \frac{|y_1|}{2^j h^{1/2}}\right)^N w(y) dy d\xi dt$$

has bound

$$\|T_{j,1} w\|_{L^2(H)} \lesssim 2^{j/2} h^{1/4} \|w\|_{L^2}.$$

We are now in the position to calculate the (t, ξ_1) integral by stationary phase. There is a stationary point when

$$\partial_t \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') = 2^j h^{-1/2} p(x, \nabla_x \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi')) = 0$$

and

$$\partial_{\eta_1} \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') = 0.$$

We need to check if this is a non-degenerate stationary point.

$$\begin{aligned}\partial_{t\eta_1}^2 \phi &= 2^j h^{-1/2} \partial_{\xi_1} p(x, \nabla_x \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi')) \\ &= 2^{-j} h^{1/2} [\nu(x, \nabla_x \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi')) (1 + O(|t|)) + O(|t|)]\end{aligned}$$

so

$$|\partial_{t\eta_1}^2 \phi| > c > 0.$$

Now since

$$\begin{aligned}\phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') &= 2^j h \langle x', \xi' \rangle + t 2^j h^{-1/2} p(2^{-j} h^{1/2} x, \nabla_x \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi')) + O(|t|^2) \\ |\partial_{\eta_1 \eta_1}^2 \phi| &\lesssim \epsilon\end{aligned}$$

Finally

$$\begin{aligned}\partial_{tt}^2 \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') &= 2^{-j} h^{1/2} \partial_t p(2^{-j} h^{1/2} x, \nabla_x \phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi')) \\ &\quad 2^{-j} h^{-1/2} [\nu O(|\epsilon|) + \sum_j 2^j h^{1/2} p_{\xi_j} \phi_{x_j, t}]\end{aligned}$$

so

$$|\partial_{tt}^2 \phi| \lesssim 1.$$

The stationary point is therefore non-degenerate. Using the stationary phase theorem we write

$$T_{j,1} w = 2^{2j(n-1)} \int e^{i2^{2j}(\langle x' - y', \xi' \rangle + \psi(x', \xi', y_1))} a(x', \xi', y) \left(1 + \frac{|y_1|}{2^j h^{1/2}}\right)^{-N} w(y) d\xi' dy$$

where if (t^c, ξ_1^c) is the critical point

$$\psi(x', \xi', y_1) = \frac{1}{2^{2j} h} (\phi(t^c, 2^{-j} h^{1/2} x', \xi_1^c, 2^{-j} h^{1/2} \xi') - y_1 \xi_1^c)$$

and

$$|D^\alpha a(x', \xi', y)| \leq C_\alpha.$$

Now we re-write this as

$$T_{j,1} w = 2^{j(n-1)} \int e^{i2^{2j}(\langle x', \xi' \rangle + \psi(x', \xi', y_1))} a(x', \xi', y) \left(1 + \frac{|y_1|}{2^j h^{1/2}}\right)^{-N} \mathcal{F}_{2^{2j}}^{n-1} w(y_1, \xi') d\xi' dy_1$$

where $\mathcal{F}_{2^{2j}}^{n-1}$ is the $n-1$ dimensional semiclassical Fourier transform, which is unitary as an operator $L_{x'}^2 \rightarrow L_{\xi'}^2$. Therefore if we freeze y_1 and set

$$(20) \quad Z_{j,1}(y_1)g = 2^{j(n-1)} \int e^{i2^{2j}(\langle x', \xi' \rangle + \psi(x', \xi', y_1))} a(x', \xi', y) g(y_1, \xi') d\xi'$$

we need to show

$$(21) \quad \|Z_{j,1}(y_1)g\|_{L^2(H)} \lesssim \|g(y_1, \cdot)\|_{L^2(\mathbb{R}^{n-1})}.$$

Providing (21) holds we may apply Holder to the integration in y_1 to obtain

$$\|T_{j,1} w\|_{L^2(H)} \lesssim 2^{j/2} h^{1/4} \|w\|_{L^2(M)}.$$

We will prove (21) via almost orthogonality. Let $x'_i, \xi'_k \in \mathbb{R}^{n-1}$ define

$$\begin{aligned}Z_{j,1}^{ik}(y_1)g &= 2^{j(n-1)} \int e^{i2^{2j}(\langle x', \xi' \rangle + \psi(x', \xi', y_1))} a(x', \xi', y) \left(1 + \frac{|y_1|}{2^j h^{1/2}}\right)^{-N} \\ &\quad \times \zeta_0(2^j |x' - x'_i|) \zeta_0(|2^j \xi' - \xi'_k|) g(y_1, \xi') d\xi'\end{aligned}$$

Note that by Young's inequality

$$\|Z_{j,1}^{ik}(y_1)\|_{L^2(\mathbb{R}^n) \rightarrow L^2(H)} \lesssim 1$$

so if we can prove these components are almost orthogonal to each other we obtain (21). Note that if $i \neq m$

$$(Z_{j,1}^{ik}(y_1))^* Z_{j,1}^{ml}(y_1) = 0$$

and if $k \neq l$

$$Z_{j,1}^{ik}(y_1)(Z_{j,1}^{ml}(y_1))^* = 0$$

so we need only check

$$(Z_{j,1}^{ik}(y_1))^* Z_{j,1}^{il}(y_1)$$

and

$$Z_{j,1}^{ik}(y_1)(Z_{j,1}^{mk}(y_1))^*$$

Now

$$\begin{aligned} (Z_{j,1}^{ik}(y_1))^* Z_{j,1}^{il}(y_1) &= 2^{2j(n-1)} \int e^{i2^{2j}(\langle x', \xi' - \eta' \rangle + \psi(x', \xi', y_1) - \psi(x', \eta', y_1))} \zeta_0(2^j |x' - x'_i|) \zeta_0(2^j |\xi' - \xi'_k|) \\ &\quad \times \zeta_0(2^j |\eta' - \xi'_l|) \tilde{a}(x', \xi', \eta', y_1) g(y_1, \eta') d\eta d\xi \end{aligned}$$

Now

$$|\nabla_{x'} \psi(x', \xi', y_1) - \psi(x', \eta', y_1)| \leq \sup_{p,q} \partial_{x'_p \xi'_q}^2 \psi |\xi' - \eta'| \leq |\xi' - \eta'|$$

due to the scaling of $2^j h^{1/2}$ on both x' and ξ' . Therefore integration by parts picks up a factor of

$$\frac{1}{2^{2j} |\xi' - \eta'|}$$

each time but loses a factor of 2^j from hitting the symbols. Therefore each iteration of integration by parts gives

$$\frac{1}{2^j |\xi' - \eta'|}$$

which ensures almost orthogonality. The same process gives almost orthogonality for $Z_{j,1}^{ik}(y_1)(Z_{j,1}^{mk}(y_1))^*$ by integrating in ξ' .

We now need to prove the same thing for $W_{j,2}$, fortunately the proof is almost identical. We have

$$W_{j,2} = 2^{-j} h^{-3/2} \int \int_0^t \chi \left(2^{-j} h^{-1/2} t \right) \nu_j(x, hD) U(t-s) E[u] ds dt$$

If we change variables $s = t - s$ we have

$$W_{j,2} = 2^{-j} h^{-3/2} \int \int_t^0 \chi \left(2^{-j} h^{-1/2} t \right) \nu_j(x, hD) U(s) E[u] ds dt$$

If we calculate the (s, ξ_1) integral by stationary phase as in the case of $W_{j,1}$ we have to account for the boundary terms. These terms however automatically pick up an extra factor of h and so (after scaling) yield operators of the form

$$\tilde{T}_{j,2} v = 2^{2j(n-1)} \cdot 2^{-j} h^{-1/2} \int e^{\frac{i}{h}(\phi(0, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') - y_1 \xi_1 + 2^{2j} h \langle y', \xi' \rangle)} \tilde{B}(t, x', \xi) v(y) dy d\xi$$

and

$$\overline{T}_{j,2} v = 2^{2j(n-1)} \cdot 2^{-j} h^{-1/2} \int e^{\frac{i}{h}(\phi(t, 2^j h^{1/2} x', \xi_1, 2^j h^{1/2} \xi') - y_1 \xi_1 + 2^{2j} h \langle y', \xi' \rangle)} \bar{B}(t, x', \xi) v(y) dy d\xi.$$

We can apply the same almost orthogonality arguments to these operators and since the symbols $\tilde{B}(t, x', \xi)$ and $\bar{B}(t, x', \xi)$ are supported in regions $|t| < 2^j h^{1/2}$ we obtain

$$\|\tilde{T}_{j,2}v\|_{L^2(H)} \lesssim 2^{j/2} h^{1/4} \|v\|_{L^2(M)}$$

and

$$\|\bar{T}_{j,2}v\|_{L^2(H)} \lesssim 2^{j/2} h^{1/4} \|v\|_{L^2(M)}$$

thereby ensuring

$$\|W_{j,2}u\|_{L^2(M)} \lesssim 2^{j/2} h^{1/4} \|v\|_{L^2(M)}.$$

□

We now need to obtain estimates for $\nu_0(x, hD)u$ here we will not be able to evaluate a stationary point. We will however be able to get enough decay from simple integration by parts.

Proposition 1.2. *If $W_{0,1}$ and $W_{0,2}$ are given by (13) and (14) then*

$$(22) \quad \|W_{0,i}u\|_{L^2(H)} \lesssim h^{1/4} \|u\|_{L^2(M)} \quad i = 1, 2.$$

Proof. In this case we can no longer calculate the (t, ξ_1) integrals via stationary phase however as $\nu(x, \xi)$ is very small we are able to prove Proposition 1.2 simply by integration by parts. Working first with $W_{0,1}$ we still scale as in the proof of Proposition 1.1 and so we need to prove that if

$$(23) \quad T_{0,1}w = h^{-3/2} \int e^{\frac{i}{h}(\phi(t, h^{1/2}x', \xi_1, h^{1/2}\xi') - y_1\xi_1) - \langle y', \xi' \rangle} B_0(t, x', \xi) \\ \nu(x, \nabla_x \phi) \left(1 + \frac{|y_1|}{h^{1/2}}\right)^N w(y) dy d\xi dt$$

then $T_{0,1}$ obeys the bounds

$$(24) \quad \|T_{0,1}w\|_{L^2(H)} \lesssim \|w\|_{L^2(M)}$$

As in the proof of Proposition 1.1 we have that

$$|\partial_{tt}^2 \phi| \leq c$$

so we can integrate by parts to obtain

$$T_{0,1}w = h^{-3/2} \int e^{\frac{i}{h}(\phi(t, h^{1/2}x', \xi_1, h^{1/2}\xi') - y_1\xi_1) - i\langle y', \xi' \rangle} B_j(t, x', \xi) \\ \times \nu(h^{1/2}x, \nabla_x \phi) \left(1 + \frac{|\phi_t|}{h^{1/2}}\right)^{-N} \left(1 + \frac{|y_1|}{h^{1/2}}\right)^{-N} w dy d\xi dt.$$

Note that the scaling on x' and ξ' mean that derivatives on the symbol are bounded $O(1)$. So we can again reduce via the Fourier transform and carry out the almost orthogonality argument as in the case $j \geq 1$. So we need to estimate the norm of one $Z_{0,1}^{ik}(y_1)$

$$Z_{0,1}^{ik}(y_1) = h^{-3/2} \int e^{\frac{i}{h}(\phi(t, h^{1/2}x', \xi_1, h^{1/2}\xi') - y_1\xi_1)} B_j(t, x', \xi) \left(1 + \frac{|\phi_t|}{h^{1/2}}\right)^{-N} \left(1 + \frac{|y_1|}{h^{1/2}}\right)^{-N} \\ \times \nu(h^{1/2}x, \nabla_x \phi) \zeta_0(|x' - x'_i|) \zeta_0(|\xi' - \xi'_k|) g(y_1, \xi') d\xi$$

Now we introduce a change of variable $\bar{\xi}' = h^{1/2} \nabla_{x'} \phi(t, h^{1/2} x, \xi_1, h^{1/2} \xi')$ Now since

$$\nabla_{x'} \phi(t, x', \xi) = \xi' + O(|t|)$$

the Jacobian of this charge of variables is $O(1)$. Then let

$$\bar{\xi}_1 = \partial_t \phi = p(h^{1/2} x', \partial_{x_1} \phi, \bar{\xi}')$$

So

$$\frac{\bar{\xi}_1}{\partial \xi_1} = h^{-1/2} \nu(h^{1/2}, \partial_{x_1} \phi, \bar{\xi}') [1 + O(|t|)]$$

So the factor

$$\left(1 + \frac{|\partial_t \phi|}{h^{1/2}}\right)^{-N}$$

becomes

$$\left(1 + \frac{|\bar{\xi}_1|}{h^{1/2}}\right)^{-N}$$

and we gain a factor of $h^{1/2}$ from the change of variables. Putting this we the almost orthogonality we have that

$$\|T_{1,0} w\|_{L^2(H)} \lesssim h^{1/4} \|w\|_{L^2(M)}$$

as required. We treat the $W_{0,2}$ term in the same fashion accounting for boundary terms as we did in the proof of Proposition 1.1. \square

It therefore only remains to verify that the terms given by E_j are small enough.

Lemma 1.3. *If E_j is given by (17) then*

$$(25) \quad \|E_j u\|_{L^2(H)} \lesssim \|u\|_{L^2(M)}.$$

Proof. We write

$$E_j v = \frac{h}{(2\pi h)^{n/2}} \int e^{\frac{i}{h} \phi(t, x, \xi)} \tilde{b}(t, x, \xi, h) \chi(t) \mathcal{F}_h v d\xi dt + O(h^\infty)$$

where $\mathcal{F}_h v$ is the semiclassical Fourier transform which is unitary. Therefore freezing t and ξ_1 we may focus on the operator $Z_j(t, \xi_1) : \mathbb{R}^n \rightarrow H$,

$$Z_j(t) g = \frac{h}{(2\pi h)^{n/2}} \int e^{\frac{i}{h} \phi(t, x', \xi_1, \xi')} \tilde{b}(t, x', \xi, h) g(\xi_1, \xi') d\xi'$$

we will show that

$$(26) \quad \|Z_j(t, \xi_1) g\|_{L^2(H)} \lesssim h^{1/2} \|g\|_{L^2(\mathbb{R}^n)}$$

with uniform constants in (t, ξ_1) . This then implies

$$\|E_j(t) v\|_{L^2(H)} \lesssim h^{1/2} \|v\|_{L^2(\mathbb{R}^n)}$$

as required.

We prove (26) via an almost orthogonality argument. Let $x'_i, \xi'_k \in \mathbb{R}^{n-1}$ be a sets of $h^{1/2}$ spaced points. We write

$$Z_j(t, \xi_1) g = \sum_{i,k} Z_j^{i,k}(t, \xi_1) g$$

$$Z_j^{i,k}(t, \xi_1) g = \frac{h}{(2\pi h)^{n/2}} \int e^{\frac{i}{h} \phi(t, x', \xi_1, \xi')} \zeta(h^{-1/2} |\xi - \xi'_k|) \zeta(h^{-1/2} |x - x'_i|) \tilde{b}(t, x', \xi) g(\xi_1, \xi') d\xi'$$

we will show that

$$(27) \quad \left\| Z_j^{i,k}(t, \xi_1)(Z_j^{l,m}(t, \xi_1))^* v \right\|_{L^2(H)} \lesssim h(1 + |i - l|)^{-N}(1 + |k - m|)^{-N} \|v\|_{L^2(H)}$$

and

$$(28) \quad \left\| (Z_j^{i,k}(t, \xi_1))^* Z_j^{l,m}(t, \xi_1) g \right\|_{L^2(H)} \lesssim h(1 + |i - l|)^{-N}(1 + |k - m|)^{-N} \|g\|_{L^2(\mathbb{R}^n)}$$

First we prove (27). Now if $k \neq m$,

$$Z_j^{i,k}(t, \xi_1)(Z_j^{l,m}(t, \xi_1))^* v = 0$$

so we may assume $k = m$ and need to estimate

$$Z_j^{i,k}(t, \xi_1)(Z_j^{l,k}(t, \xi_1))^* v = \int K^{i,l,k}(t, \xi_1, x', y') v(y') dy'$$

$$K(t, \xi_1, x', y') = \frac{h^2}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi(t, x', \xi_1, \xi') - \phi(t, y', \xi_1, \xi'))} B^{i,l,k}(t, x', y', \xi) d\xi'$$

where

$$B^{i,l,k}(t, x', y', \xi) = \zeta(h^{-1/2}|x' - x'_i|)\zeta(h^{-1/2}|y' - x'_i|)\zeta(h^{-1/2}|\xi' - \xi'_k|)d(t, x', y', \xi)$$

$$|D^\alpha d(t, x', y', \xi)| \leq C_\alpha h^{-|\alpha|/2}$$

Note that

$$\phi(t, x', \xi_1, \xi') - \phi(t, y', \xi_1, \xi') = \langle x' - y', \xi' \rangle + O(|x - y|^2)$$

so

$$|\nabla_{\xi'}(\phi(t, x', \xi) - \phi(t, y', \xi))| > c|x' - y'|$$

if we integrate by parts in ξ' we gain a factor of

$$\frac{h}{|x' - y'|}$$

for each iteration but loose up to a factor of $h^{-1/2}$ from hitting the symbol. Therefore

$$|K(t, \xi_1, x', y')| \lesssim h^2 \cdot h^{-n} h^{(n-1)/2} \zeta(h^{-1/2}|x' - x'_i|)\zeta(h^{-1/2}|y' - x'_i|)$$

and

$$\left\| Z_j^{i,k}(t, \xi_1)(Z_j^{l,k}(t, \xi_1))^* v \right\|_{L^2(H)} \lesssim h(1 + |i - l|)^{-N} \|v\|_{L^2(H)}.$$

We can repeat the same argument (this time integrating by parts in x' to show that (28) is also true. Therefore by Cotlar-Stein almost orthogonality

$$\|Z_j v\|_{L^2(H)} \lesssim h^{1/2} \|v\|_{L^2(M)}$$

as required. \square

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